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Transient flow of a viscous compressible fluid in a circular tube after a sudden point impulse

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The flow of a viscous compressible fluid in a circular tube generated by a sudden impulse at a point on the axis is studied on the basis of the linearized Navier–Stokes equations. A no-slip boundary condition is assumed to hold on the wall of the tube. Owing to the finite velocity of sound the flow behaviour differs qualitatively from that of an incompressible fluid. The flow velocity and the pressure disturbance at any fixed point different from the source point vanish at short time and decay at long times with a $t^{-3/2}$ power law.

1. Introduction

Compressibility has a significant effect on the dynamics of flow of a viscous fluid in confined geometry. Owing to the finite velocity of sound the local disturbance of quiescent fluid caused by a sudden impulse at a chosen distant point occurs only after a finite travelling time, rather than instantaneously, as in an incompressible fluid. It has been argued by Hagen *et al.* (1997) that in a slit or a tube there is also a significant after effect at long times, different from that in an incompressible fluid. Their computer simulation showed that the velocity correlation function of a suspended particle decays at long times with a negative algebraic tail, in contrast with the behaviour in infinite space, where the long-time tail has a positive amplitude. It was later shown that this is due to the coupling to diffusive sound modes of long wavelength (Pagonabarraga *et al.* 1999).

In the following we study the dynamics of flow in a circular tube, generated by a sudden point impulse on the axis of the tube and in the direction of the axis. The analysis is based on the solution of the linearized Navier–Stokes equations for a compressible viscous fluid. The explicit form of the corresponding Green function is found as an integral over wavenumber and frequency. The calculation is a generalization of an earlier one for the same situation in an incompressible fluid (Felderhof 2009). In a compressible fluid the component of the flow velocity in the direction of the axis at any fixed point decays in time with a negative algebraic $t^{-3/2}$ long-time tail. Similarly the pressure disturbance at any point also decays in time with a $t^{-3/2}$ long-time tail. We derive analytic expressions for the amplitude of the long-time tails. The amplitude is inversely proportional to the velocity of sound.

The velocity autocorrelation function of a Brownian particle is related to the frequency-dependent response of the particle to an applied oscillatory force by the

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fluctuation-dissipation theorem (Kubo, Toda & Hashitsume 1991). The frequency-dependent admittance of a particle located on the axis can be calculated from the Green function and from a generalized Faxén theorem derived by Bedeaux & Mazur (1974) in point approximation, i.e. to first order in the ratio a/b, where a is the particle radius and b is the radius of the tube. The velocity autocorrelation function calculated in this manner is found to decay with a negative $t^{-3/2}$ long-time tail in agreement with computer simulation (Hagen *et al.* 1997; Pagonabarraga *et al.* 1999). At shorter times the calculated time-dependent correlation function agrees only qualitatively with the result of computer simulation because the point approximation used in the calculation has limited validity for the relatively large particle used in the simulation.

The integral of the Green function over the entire time is identical with the steady-state Green function studied by Hasimoto (1976) and Liron & Shahar (1978) for an incompressible fluid. Therefore the integral of the flow pattern over the entire time shows the complicated sequence of eddies found by these authors and by Blake (1979).

2. Linear hydrodynamics of flow in a circular tube

We consider a viscous compressible fluid of shear viscosity η, bulk viscosity η_v and equilibrium mass density ρ_0 located in a circular tube of radius b. We choose coordinates such that the z axis is along the axis of the tube, and we use cylindrical coordinates (R, φ, z). For time $t<0$ the fluid is at rest at static pressure p_s. At time $t=0$ an impulse $\boldsymbol{P}$ is imparted to the fluid at the origin and directed along the z axis. We study the resulting motion of the fluid for time $t>0$.

For small-amplitude motion the flow velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ and the pressure $p(\boldsymbol{r}, t)$ are governed by the linearized Navier–Stokes equations,

$$\left.\begin{aligned} \rho_0\frac{\partial \boldsymbol{v}}{\partial t} &= \eta\nabla^2\boldsymbol{v} + \left(\frac{1}{3}\eta + \eta_v\right)\nabla\nabla\cdot\boldsymbol{v} - \nabla p + \boldsymbol{P}\delta(\boldsymbol{r})\delta(t), \\ \frac{\partial p}{\partial t} &= -\rho_0 c_0^2\nabla\cdot\boldsymbol{v}, \end{aligned}\right\} \tag{2.1}$$

with impulse $\boldsymbol{P}=P\boldsymbol{e}_z$ and long-wave sound velocity c_0. We assume that the flow velocity satisfies the no-slip boundary condition at the wall of the cylinder, i.e. $\boldsymbol{v}=\boldsymbol{0}$ at $R=b$. We look for the solution of (2.1) for which the flow velocity $\boldsymbol{v}(\boldsymbol{r}, t)$ vanishes and the pressure tends to the static pressure p_s as $z\to\pm\infty$ at any time t.

After Fourier analysis in time we find that the equations for the Fourier components

$$\boldsymbol{v}_\omega(\boldsymbol{r}) = \int_0^\infty e^{i\omega t}\boldsymbol{v}(\boldsymbol{r}, t)\,dt, \qquad p_\omega(\boldsymbol{r}) = \int_0^\infty e^{i\omega t}[p(\boldsymbol{r}, t) - p_s]\,dt \tag{2.2}$$

are

$$\left.\begin{aligned} \eta\left(\nabla^2\boldsymbol{v}_\omega - \alpha^2\boldsymbol{v}_\omega\right) + \left(\frac{1}{3}\eta + \eta_v\right)\nabla\nabla\cdot\boldsymbol{v}_\omega - \nabla p_\omega &= -\boldsymbol{P}\delta(\boldsymbol{r}), \\ \nabla\cdot\boldsymbol{v}_\omega - i\beta p_\omega &= 0, \end{aligned}\right\} \tag{2.3}$$

where we have used the abbreviations

$$\alpha = \sqrt{\frac{-i\omega\rho_0}{\eta}}, \qquad \mathrm{Re}\,\alpha > 0, \qquad \beta = \frac{\omega}{\rho_0 c_0^2}. \tag{2.4}$$

We write the Fourier-transformed flow velocity as

$$\boldsymbol{v}_{\omega}(\boldsymbol{r}) = \boldsymbol{v}_{0\omega}(\boldsymbol{r}) + \boldsymbol{v}_{1\omega}(\boldsymbol{r}), \tag{2.5}$$

where $\boldsymbol{v}_{0\omega}(\boldsymbol{r})$ is the solution for infinite space and $\boldsymbol{v}_{1\omega}(\boldsymbol{r})$ is the reflected flow because of the presence of the boundary. The flows can be expressed as

$$\boldsymbol{v}_{0\omega}(\boldsymbol{r}) = \mathsf{G}_0(\boldsymbol{r} - \boldsymbol{r}_0)\cdot\boldsymbol{P}, \quad \boldsymbol{v}_{\omega}(\boldsymbol{r}) = \mathsf{G}(\boldsymbol{r}, \boldsymbol{r}_0)\cdot\boldsymbol{P}, \tag{2.6}$$

with Green functions G_0 and G. The Green function for infinite space is translationally invariant and has been given explicitly by Jones 1981,

$$\mathsf{G}_0(\boldsymbol{r}) = \frac{1}{4\pi\eta}\left(\frac{\mathrm{e}^{-\alpha r}}{r}\mathbf{1} + \alpha^{-2}\nabla\nabla\frac{\mathrm{e}^{\mathrm{i}\mu r} - \mathrm{e}^{-\alpha r}}{r}\right), \tag{2.7}$$

with the abbreviation

$$\mu = \omega/c, \quad \mathrm{Im}\mu > 0, \tag{2.8}$$

where

$$c = c_0\left[1 - \mathrm{i}\beta\left(\frac{4}{3}\eta + \eta_v\right)\right]^{1/2}. \tag{2.9}$$

Since we consider $\boldsymbol{P} = P\boldsymbol{e}_z$ the azimuthal component of the flow velocity vanishes on account of axial symmetry. The two non-vanishing components of the flow velocity $\boldsymbol{v}_{0\omega}$ for infinite space can be expressed as

$$\left.\begin{aligned} v_{0R\omega}(\boldsymbol{r}) &= \frac{P}{2\pi^2\eta\alpha^2}\int_0^\infty \hat{v}_{0R}(k,\omega,R)\sin kz\,\mathrm{d}k, \\ v_{0z\omega}(\boldsymbol{r}) &= \frac{P}{2\pi^2\eta\alpha^2}\int_0^\infty \hat{v}_{0z}(k,\omega,R)\cos kz\,\mathrm{d}k, \end{aligned}\right\} \tag{2.10}$$

with amplitudes

$$\left.\begin{aligned} \hat{v}_{0R}(k,\omega,R) &= -ksK_1(sR) + kuK_1(uR), \\ \hat{v}_{0z}(k,\omega,R) &= s^2K_0(sR) - k^2K_0(uR), \end{aligned}\right\} \tag{2.11}$$

with modified Bessel functions $K_n(x)$ and the abbreviations

$$s = \sqrt{k^2 + \alpha^2}, \quad u = \sqrt{k^2 - \mu^2}. \tag{2.12}$$

In analogy to (2.10) we write the reflected flow velocity $\boldsymbol{v}_{1\omega}$ as

$$\left.\begin{aligned} v_{1R\omega}(\boldsymbol{r}) &= \frac{P}{2\pi^2\eta\alpha^2}\int_0^\infty \hat{v}_{1R}(k,\omega,R)\sin kz\,\mathrm{d}k, \\ v_{1z\omega}(\boldsymbol{r}) &= \frac{P}{2\pi^2\eta\alpha^2}\int_0^\infty \hat{v}_{1z}(k,\omega,R)\cos kz\,\mathrm{d}k, \end{aligned}\right\} \tag{2.13}$$

with amplitudes

$$\left.\begin{aligned} \hat{v}_{1R}(k,\omega,R) &= A(k,\omega)\hat{v}_{Rp}(k,\omega,R) + B(k,\omega)\hat{v}_{Rv}(k,\omega,R), \\ \hat{v}_{1z}(k,\omega,R) &= A(k,\omega)\hat{v}_{zp}(k,\omega,R) + B(k,\omega)\hat{v}_{zv}(k,\omega,R), \end{aligned}\right\} \tag{2.14}$$

where

$$\left.\begin{aligned} \hat{v}_{Rp}(k,\omega,R) &= kuI_1(uR), \quad \hat{v}_{Rv}(k,\omega,R) = ksI_1(sR), \\ \hat{v}_{zp}(k,\omega,R) &= k^2I_0(uR), \quad \hat{v}_{zv}(k,\omega,R) = s^2I_0(sR), \end{aligned}\right\} \tag{2.15}$$

with modified Bessel functions $I_n(x)$. Together with the expression for the pressure given below in (2.21) the expressions in (2.13) provide the general solution of (2.3), which is regular everywhere, except infinity. This must be combined with the singular solution given by (2.10), (2.11) and (2.20) below. From the no-slip boundary condition at $R=b$ we find for the coefficients $A(k,\omega)$ and $B(k,\omega)$,

$$A(k,\omega)=\frac{P(k,\omega)}{Z(k,\omega)},\quad B(k,\omega)=\frac{Q(k,\omega)}{Z(k,\omega)} \tag{2.16}$$

with denominator

$$Z(k,\omega)=bs\left[usI_0(sb)I_1(ub)-k^2I_0(ub)I_1(sb)\right] \tag{2.17}$$

and numerators

$$\left.\begin{aligned}P(k,\omega)&=s^2-sk^2bK_0(ub)I_1(sb)-s^2ubI_0(sb)K_1(ub),\\ Q(k,\omega)&=k^2-s^2ubK_0(sb)I_1(ub)-k^2sbI_0(ub)K_1(sb).\end{aligned}\right\} \tag{2.18}$$

The Fourier transform of the pressure corresponding to the point excitation in infinite space takes the form

$$p_{0\omega}(\boldsymbol{r})=\frac{P}{4\pi}\frac{c_0^2}{c^2}\frac{z}{r^3}(1-\mathrm{i}\mu r)\mathrm{e}^{\mathrm{i}\mu r}. \tag{2.19}$$

This can be cast in the alternative form,

$$p_{0\omega}(\boldsymbol{r})=\frac{P}{2\pi^2}\frac{c_0^2}{c^2}\int_0^\infty K_0(uR)k\sin kz\,\mathrm{d}k. \tag{2.20}$$

The pressure corresponding to the reflected flow field given by (2.13) is given by

$$p_{1\omega}(\boldsymbol{r})=\frac{P}{2\pi^2}\frac{c_0^2}{c^2}\int_0^\infty A(k,\omega)I_0(uR)k\sin kz\,\mathrm{d}k. \tag{2.21}$$

At zero frequency $A(k,0)=2A_2(k)$ with coefficient $A_2(k)$ given by (3.4) of Felderhof (2009), so that in the zero-frequency limit (2.21) reduces to the steady-state pressure given by (3.13) of Felderhof (2009).

It is evident from (2.3) that in the limit of zero frequency the equations reduce to the steady-state Stokes equations for an incompressible fluid. Hence the integral over the entire time of the velocity field $\boldsymbol{v}(\boldsymbol{r},t)$ and of the pressure disturbance $p(\boldsymbol{r},t)-p_s$ reduces to the expressions found by Hasimoto (1976), Liron & Shahar (1978) and Blake (1979). The steady-state flow velocity shows an infinite sequence of vortex rings and decays to zero as $z\to\pm\infty$. The steady-state pressure disturbance tends to $\pm$ a constant as $z\to\pm\infty$.

3. Pressure and flow velocity

The explicit expressions found above allow calculation of velocity and pressure at any point $\boldsymbol{r}$ in the tube and at any time t by inversion of the Fourier transform with respect to frequency. We consider first the pressure, since the expressions are somewhat simpler than for the velocity. The pressure may be written as

$$p(\boldsymbol{r},t)=p_s+\delta p(\boldsymbol{r},t), \tag{3.1}$$

where the time dependence of the disturbance δp follows by inverse transform of the expressions in (2.19) and (2.21). In the incompressible limit a non-vanishing pressure perturbation is established everywhere instantaneously because of the infinite velocity of sound, and the pressure surge at a chosen point diverges as $1/\sqrt{t}$ at short

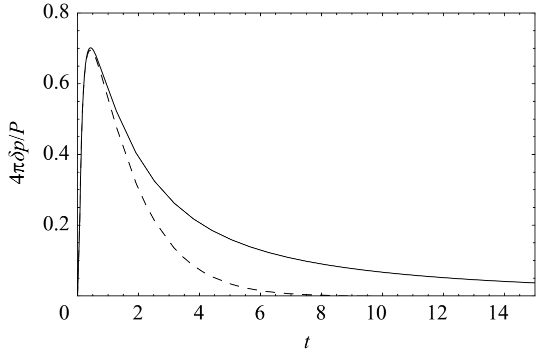

FIGURE 1. Plot of the normalized pressure disturbance $4\pi\delta p/P$ at the point $R=0.5b$, $z=0.5b$ in a tube of radius $b=1$ in a compressible fluid with properties as in the computer simulation of Hagen *et al.* (1997) as a function of time (solid curve). We compare with the corresponding disturbance in infinite space (dashed curve).

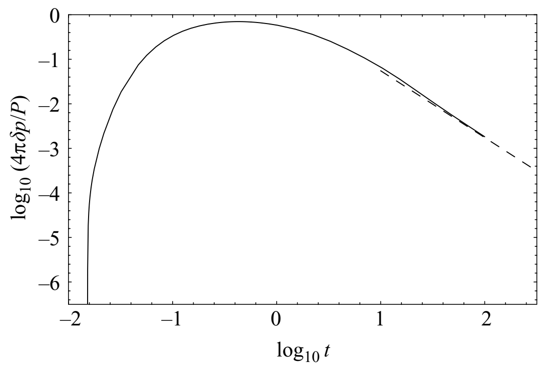

FIGURE 2. Plot of $\log_{10}[4\pi\delta p/P]$ as a function of $\log_{10} t$ corresponding to figure 1 (solid curve). We compare with the straight line corresponding to the long-time tail given by (3.5) (dashed line).

time, before decaying with a $t^{-5/2}$ power law at long times (Felderhof 2009). For a compressible fluid the pressure at any point different from the origin initially equals the static pressure p_s. At later times a pressure pulse passes through the point. We shall find that the confinement in the tube causes a slow decay of the pressure disturbance with a $t^{-3/2}$ power law at long times.

We consider a compressible fluid with parameters chosen as in the computer simulation of Hagen *et al.* (1997). In their units the mass density ρ_0 is 24; the shear viscosity η is 14.4; and the sound velocity c_0 is $1/\sqrt{2}$. The bulk viscosity $\eta_v=1/30$ (Frenkel & Lowe 2005, private communication). In figure 1 we show the normalized pressure perturbation $4\pi\delta p/P$ at the chosen point $R=0.5b$, $z=0.5b$ for a tube of radius $b=1$ as a function of time. We compare with the corresponding pulse for infinite space. It is evident that at short time the two pressure pulses are the same but that the confinement in the tube causes a slow decay at long times. The slow decay is due to the contribution $p_1(\boldsymbol{r},t)$. In figure 2 we show the plot of $\log_{10}(4\pi\delta p/P)$ for the same fluid at the same point, as a function of $\log_{10} t$, making evident the algebraic tail at long times.

In order to calculate the amplitude of the long-time tail we consider the Fourier transform in (2.21). For small wavenumber k the denominator of the coefficient $A(k, \omega)$, given by $Z(k, \omega)$ in (2.17), has a zero on the negative imaginary ω-axis given approximately by

$$\omega_d = -\mathrm{i}D^* k^2, \tag{3.2}$$

corresponding to a non-propagating sound wave with effective diffusion coefficient

$$D^* = \frac{c_0^2 \rho_0 b^2}{8\eta}. \tag{3.3}$$

The difference with expression (14) of Pagonabarraga *et al.* (1999) is caused by the difference in geometry. By use of the identity

$$\int_0^\infty \mathrm{e}^{-D^* k^2 t} k^2 \, \mathrm{d}k = \frac{\sqrt{\pi}}{4(D^* t)^{3/2}} \tag{3.4}$$

we see that a superposition of such diffusive waves causes a $t^{-3/2}$ long-time tail. From the residue of the pole of $Z(k, \omega)$ at $\omega_d = -\mathrm{i}D^* k^2$ in the complex ω-plane we find the long-time behaviour

$$p_1(\boldsymbol{r}, t) \approx \frac{\sqrt{2\nu}}{c_0 b^3} \frac{z}{(\pi t)^{3/2}} P \quad \text{as } t \to \infty, \tag{3.5}$$

where $\nu = \eta/\rho_0$ is the kinematic viscosity. The pressure disturbance $\delta p(\boldsymbol{r}, t)$ has the same long-time behaviour.

In a compressible fluid the flow velocity at any point different from the origin vanishes initially, since the finite velocity of sound limits the speed of propagation of the disturbance because of the initial impulse. It follows from the linearized Navier–Stokes equations (2.1) that the long-time behaviour of the pressure, given by (3.5), corresponds to a similar $t^{-3/2}$ long-time tail in the velocity component $v_z(\boldsymbol{r}, t)$. Mathematically this corresponds to the simple pole in the expressions for the coefficients $A(k, \omega)$ and $B(k, \omega)$ for small k on the negative imaginary ω-axis. It follows from the expansion of expression (2.7) for the infinite space Green function that the velocity component $v_{0z}(\boldsymbol{r}, t)$ shows the same positive $t^{-3/2}$ long-time tail because of viscous diffusion of momentum as for an incompressible fluid. This is cancelled by a viscous contribution to $v_{1z}(\boldsymbol{r}, t)$ by reflection at the wall of the tube, in the same way as for an incompressible fluid. The remaining long-time tail is due to the simple-pole contribution, corresponding to the non-propagating sound wave, and can be calculated as above. It follows by comparison with the numerical calculation of the inverse transform of the expressions in § 2 that this picture is correct.

For the long-time part of the velocity component $v_{1z}(\boldsymbol{r}, t)$ we find

$$v_{1z}(\boldsymbol{r}, t) \approx -\frac{P}{12\rho_0(\pi\nu t)^{3/2}} - \frac{\sqrt{2\nu}}{4\eta c_0 b}\left(1 - \frac{R^2}{b^2}\right)\frac{P}{(\pi t)^{3/2}} \quad \text{as } t \to \infty, \tag{3.6}$$

where the first term cancels the long-term flow for infinite space, and the second term is the pole contribution from the diffusive sound wave. The first term is confirmed by an asymptotic calculation for small wavenumber of the integral for v_{1z} in the incompressible limit. Together with (3.5) the asymptotic flow velocity satisfies the linearized Navier–Stokes equations (2.1), as well as the no-slip boundary condition on the wall of the tube. The asymptotic flow is independent of the bulk viscosity η_v, but the complete solution does depend on this transport coefficient.

4. Velocity relaxation and Brownian motion

In conclusion we consider the velocity autocorrelation function of a Brownian particle of radius a and mass m_p, initially located at the origin. We consider only the z-component of the motion. The velocity autocorrelation function may be evaluated as the Fourier transform of the zz-component of the frequency-dependent admittance tensor, which gives the mean velocity response of the particle to an applied harmonic force. The admittance tensor differs from that for infinite space because of the no-slip boundary condition at the wall. For $a \ll b$ the difference may be expressed in terms of a reaction field tensor. Here we need the zz-component given by

$$F_{zz}(\mathbf{0}, \omega) = v_{1z\omega}(\mathbf{0})/P. \tag{4.1}$$

The relevant element of the admittance tensor is (Felderhof 2005)

$$\mathscr{Y}_{zz}(\mathbf{0}, \omega) = \mathscr{Y}_0(\omega)[1 + A(\omega)C(\omega)F_{zz}(\mathbf{0}, \omega)], \tag{4.2}$$

where $\mathscr{Y}_0(\omega)$ is the scalar admittance for infinite space

$$\mathscr{Y}_0(\omega) = [-i\omega m_p + \zeta(\omega)]^{-1} \tag{4.3}$$

with friction coefficient (Bedeaux & Mazur 1974)

$$\zeta(\omega) = \frac{4\pi}{3}\eta\alpha^2 a \frac{(9 + 9\alpha a + \alpha^2 a^2)B(\omega) + \mu^2 a^2 A(\omega)}{2\alpha^2 B(\omega) - \mu^2 A(\omega)} \tag{4.4}$$

with functions $A(\omega)$ and $B(\omega)$ given by

$$A(\omega) = 1 + \alpha a + \frac{1}{3}\alpha^2 a^2, \quad B(\omega) = 1 - \mathrm{i}\mu a - \frac{1}{3}\mu^2 a^2. \tag{4.5}$$

The factor $C(\omega)$ in (4.2) is given by

$$C(\omega) = 12\pi\eta\alpha^2 a \frac{B(\omega)}{2\alpha^2 B(\omega) - \mu^2 A(\omega)}. \tag{4.6}$$

The zero-frequency admittance is the particle mobility. This takes the form

$$\mu_{zz}(\mathbf{0}) = \frac{1}{6\pi\eta a}\left(1 - k_0\frac{a}{b}\right), \tag{4.7}$$

with coefficient $k_0 = 2.10444$, like for an incompressible fluid (Faxén 1959).

In the theory of Brownian motion the velocity autocorrelation function of the particle is defined by

$$C_{zz}(t) = \langle U_z(t)U_z(0)\rangle, \tag{4.8}$$

where the angle brackets denote the equilibrium ensemble average. According to the fluctuation-dissipation theorem its Fourier transform is given by

$$\hat{C}_{zz}(\omega) = \int_0^\infty e^{i\omega t} C_{zz}(t)\,\mathrm{d}t = k_B T \mathscr{Y}_{zz}(\mathbf{0}, \omega). \tag{4.9}$$

The reaction factor $F_{zz}(\mathbf{0}, \omega)$ in (4.2) may be regarded as the Fourier transform of a function $\psi_z(t)$ according to

$$F_{zz}(\mathbf{0}, \omega) = \frac{1}{6\pi\rho_0}\int_0^\infty \mathrm{e}^{\mathrm{i}\omega t}\psi_z(t)\,\mathrm{d}t. \tag{4.10}$$

The function $\psi_z(t)$ starts at zero, since the sound wave needs a finite time to be reflected from the wall of the tube. It follows from (3.6) that it decays with a $t^{-3/2}$

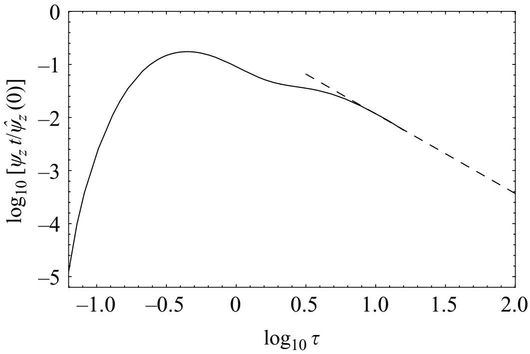

FIGURE 3. Plot of $\log_{10}[\psi_z(t)/\hat{\psi}_z(0)]$ as a function of $\log_{10}\tau$ for the same fluid as in figure 1 (solid curve). We compare with the straight line corresponding to the long-time tail given by (4.11) (dashed line).

long-time tail as

$$\psi_z(t) \approx -\frac{1}{2\sqrt{\pi}b^3}\left[1 + 3\sqrt{2}\frac{\nu}{bc_0}\right]\tau^{-3/2} \quad \text{as } t \to \infty, \tag{4.11}$$

where $\tau = t/\tau_b$ with $\tau_b = b^2/\nu$. In figure 3, we plot the ratio $\psi_z(t)/\hat{\psi}_z(0)$, where $\hat{\psi}_z(0) = -k_0/(b\nu)$, as a function of τ on a double-logarithmic scale. The product $A(\omega)C(\omega)$ in (4.2) has the low-frequency expansion

$$A(\omega)C(\omega) = 6\pi\eta a(1 + \alpha a) + O(\omega). \tag{4.12}$$

Combining the results we find for the low-frequency expansion of the admittance

$$\mathscr{Y}_{zz}(\mathbf{0}, \omega) = \frac{1}{6\pi\eta a}\left(1 - k_0\frac{a}{b}\right)\frac{\sqrt{2}}{2\pi\eta}\frac{\nu}{bc_0}\alpha + O(\omega). \tag{4.13}$$

Using the initial value $C_{zz}(0) = k_B T/m_p$ we hence find for the long-time behaviour of the normalized velocity autocorrelation function

$$\frac{C_{zz}(t)}{C_{zz}(0)} \approx -\frac{\sqrt{2}}{4\pi^{3/2}}\frac{m_p\nu}{\rho_0 c_0 b^4}\tau^{-3/2} \quad \text{as } t \to \infty. \tag{4.14}$$

This corresponds to the second term in (3.6). In figure 4, we plot the normalized velocity autocorrelation function $C_{zz}(t)/C_{zz}(0)$, calculated from (4.9), as a function of τ for a neutrally buoyant particle of radius $a = 5b/9$ in the same fluid as before, a case studied in computer simulation by Hagen *et al.* (1997) and Pagonabarraga *et al.* (1999). In figure 5 we plot the corresponding function $\log_{10}|C_{zz}(t)/C_{zz}(0)|$ as a function of $\log_{10}\tau$. The correlation function passes through zero and decays with a $t^{-3/2}$ long-time tail of negative amplitude, as in the computer simulation. The theoretical curve and the computer simulation curve are qualitatively similar. Presumably the difference is because in the simulation the ratio a/b is not small. In the theoretical calculation the admittance is calculated under the assumption that the ratio is small. We see from (4.7) that for $a = 5b/9$ the zero-frequency admittance in the approximation used becomes negative, so that finite-size corrections are appreciable.

The plots in figures 4 and 5 should be compared with the corresponding ones for an incompressible fluid shown in figures 12 and 13 of Felderhof (2009). The comparison shows that fluid compressibility has a strong influence on the velocity autocorrelation

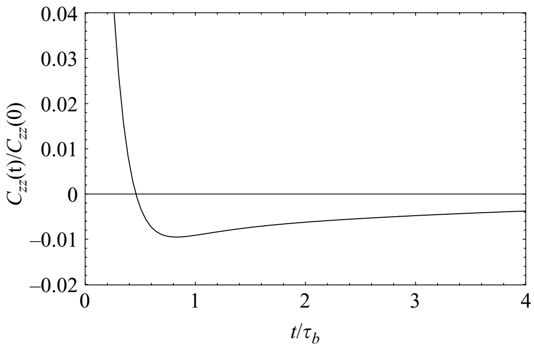

FIGURE 4. Plot of the normalized velocity autocorrelation function $C_{zz}(t)/C_{zz}(0)$ as a function of τ for a neutrally buoyant particle of radius $a = 5b/9$ centred on average at the origin in the same fluid as in figure 1.

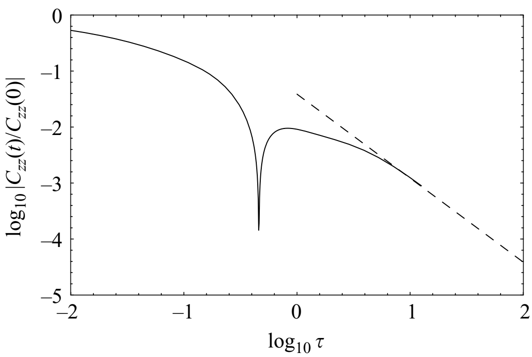

FIGURE 5. Plot of $\log_{10}|C_{zz}(t)/C_{zz}(0)|$ as a function of $\log_{10}\tau$ corresponding to figure 4 (solid curve). We compare with the straight line corresponding to the long-time tail given by (4.14) (dashed line).

function of a Brownian particle confined to a tube. In the incompressible limit the correlation function decays at long times with a $t^{-5/2}$ power law. In figure 6 we show the analogue of figure 5 for a fluid with sound velocity $c_0 = \sqrt{5}$ but otherwise the same properties as mentioned at the beginning of § 3. The fluid corresponding to figure 6 is 10 times less compressible than that corresponding to figure 5. The graph shows the crossover of the decay to the slowly varying $t^{-3/2}$ long-time tail.

5. Discussion

The above calculation of the Green function for the flow of a compressible viscous fluid confined in a circular tube shows that the flow differs significantly from that of an incompressible fluid, not only at short time but also at long times. The amplitude of the algebraic long-time tail increases strongly as the radius of the tube gets smaller. This suggests that in particular in microrheology it is important to take fluid compressibility into account in the description of dynamical flow phenomena. At zero frequency the linearized Navier–Stokes equations do not depend on compressibility, so that in the description of steady-state flow phenomena fluid compressibility is not relevant.

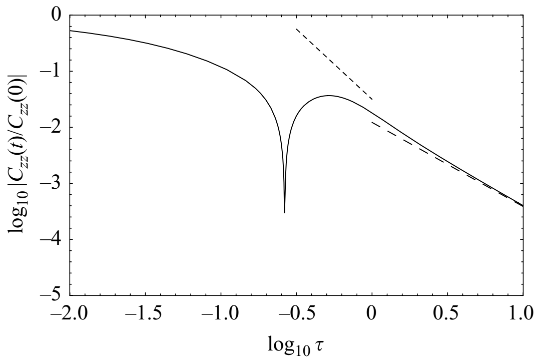

FIGURE 6. Same as figure 5, but for a fluid with sound velocity $c_0 = \sqrt{5}$, corresponding to 10 times less compressibility. We compare with the straight line corresponding to a $t^{-5/2}$ power law (short dashes).

The calculation of the flow at a particular point in space and time requires numerical inversion of a frequency-dependent Fourier transform of the Green function. The response to an oscillatory force at a particular frequency is of interest in itself. It may be used for a discussion of small-amplitude swimming or pumping in the confined geometry of a circular tube. The calculation suggests that for narrow tubes it is important to take account of compressibility in the description of these phenomena.

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